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NOTE ON $n-$GROUPS

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Abstract. Among the results of the paper is the following proposition. Let $n \geq 3$ and let (Q, A) be an $n-$grupoid. Then: (Q, A) is an $n-$group iff there are mappings $\boldsymbol{\alpha}$ and β, respectively, of the sets Q^{n-2} and Q into the set Q such that the laws

$$A(A(x_1^n), x_{n+1}^{2n-1}) = A(x_1, A(x_2^{n+1}), x_{n+2}^{2n-1}),$$

$$\beta A(x_1^n) = A(x_1^{n-1}, \beta(x_n)) = A(x_1^{n-2}, \beta(x_{n-1}), x_n),$$

$$A(x, a_1^{n-2}, \boldsymbol{\alpha}(a_1^{n-2})) = A(b_1^{n-2}, \boldsymbol{\alpha}(b_1^{n-2}), x) \text{ and}$$

$$\beta A(x, c_1^{n-2}, \boldsymbol{\alpha}(c_1^{n-2})) = x$$

hold in the algebra $(Q, \{A, \boldsymbol{\alpha}, \beta\})$ [:3.1].

1. Preliminaries

1.1. Definitions: *Let $n \geq 2$ and let (Q, A) be an $n-$groupoid. Then: (a) we say that (Q, A) is an $n-$semigroup iff for every $i, j \in \{1, ..., n\}$, $i < j$, the following law holds*

$$A(x_1^{i-1}, A(x_i^{i+n-1}), x_{i+n}^{2n-1}) = A(x_1^{j-1}, A(x_j^{j+n-1}), x_{j+n}^{2n-1})$$

[: $\langle i, j\rangle-$ *associative law];*

(b) we say that (Q, A) is an $n-$quasigroup iff for every $i \in \{1, ..., n\}$ and for every $a_1^n \in Q$ there is exactly one $x_i \in Q$ such that the following equality holds

$$A(a_1^{i-1}, x_i, a_i^{n-1}) = a_n; \quad \textit{and}$$

(c) we say that (Q, A) is a Dörnte $n-$group [briefly; $n-$group] iff (Q, A) is an $n-$semigroup and an $n-$quasigroup as well.

A notion of an $n-$group was introduced by W. Dörnte in [1] as a generalization of the notion of a group.

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1.2. Proposition [2]: *Let $n \geq 3$ and let (Q, A) be an $n-$semigroup. Further on, let i be an arbitrary element of the set $\{1, ..., n-2\}$. Then: (Q, A) is an $n-$group iff for every $a, b, c \in Q$ and for every sequence a_1^{n-3} over Q[: 1.1; $n \geq 3$] there is exactly one $\xi \in Q$ such that the following equality holds*

$$A(a, a_1^{i-1}, \xi, a_i^{n-3}, b) = c.$$

See, also [9].

1.3. Definition [6]: *Let $n \geq 2$ and let (Q, A) be an $n-$groupoid. Further on, let $\mathbf{e}$ be an mapping of the set Q^{n-2} into the set Q. Let also $\{i, j\} \subseteq \{1, ..., n\}$ and $i < j$. Then: $\mathbf{e}$ is an $\{i, j\}-$ neutral operation of the $n-$grupoid (Q, A) iff the following formula holds*

$$(\forall a_i \in Q)_1^{n-2}(\forall x \in Q)(A(a_1^{i-1}, \mathbf{e}(a^{n-2}), a_i^{j-2}, x, a_{j-1}^{n-2}) = x$$
$$\wedge A(a_1^{i-1}, x, a_i^{j-2}, \mathbf{e}(a_1^{n-2}), a_{j-1}^{n-2}) = x).$$

[For $n = 2$, $\mathbf{e}(a_1^{n-2}) = \mathbf{e}(\emptyset) \in Q$ is a neutral element of the groupoid (Q, A).]

1.4. Proposition [6]: *Let $n \geq 2$, $\{i, j\} \subseteq \{1, ..., n\}$ and $i < j$. Then in every $n-$groupoid there is at most one $\{i, j\}-$neutral operation.*

1.5. Proposition [6]: *In every $n-$group, $n \geq 2$, there is a $\{1, n\}-$ neutral operation.*

[There are $n-$groups without $\{i, j\}-$ neutral operations with $\{i, j\} \neq \{1, n\}$ [: [7]]. In [7], $n-$groups with $\{i, j\}-$neutral operations, for $\{i, j\} \neq \{1, n\}$ are described.]

1.6. Proposition [6]: *Let $n \geq 3$ and let (Q, A) be an $n-$semigroup. Then: (Q, A) is an $n-$group iff (Q, A) has a $\{1, n\}-$neutral operation.*

1.7. Remark: In [8] it was shown that the condition "...(Q, A) as an $n-$semigroup ..." can be weakened to the condition "...(Q, A) is an $\langle 1, 2\rangle-$ associative $n-$grupoid ..." or to the condition "...(Q, A) is an $\langle n-1, n\rangle-$ associative $n-$groupoid ...".

1.8. Proposition: *Let $n \geq 3$ and let (Q, A) be an $n-$groupoid. Then the following statements holds:*

(I) If (a) the $\langle 1, 2\rangle-$associative law holds in (Q, A), and (b) for every x, y, $a_1^{n-1} \in Q$ the following implication holds

$$A(x, a_1^{n-1}) = A(y, a_1^{n-1}) \Rightarrow x = y,$$

then (Q, A) is an $n-$semigroup; and

(II) If ($\overline{a}$) the $\langle n-1, n\rangle-$associative law holds in (Q, A), and ($\overline{b}$) for every $x, y, a_1^{n-1} \in Q$ the following implication holds

$$A(a_1^{n-1}, x) = A(a_1^{n-1}, y) \Rightarrow x = y,$$

then (Q,A) *is an* $n-$*semigroup.*

Proposition 1.8 can be proved by the method of E.I. Sokolov from [3].

2. Auxiliary propositions

2.1. Proposition: *Let* (Q,A) *an be* $n-$*group,* $\mathbf{e}$ *its* $\{1,n\}-$*neutral operation and* $n \geq 2$. *Then for every sequence* a_1^{n-2} *over* Q, *every sequence* b_1^{n-2} *over* Q *and for every* $x \in Q$ *the following equalities hold*

(i) $A(\mathbf{e}(a_1^{n-2}), a_1^{n-2}, x) = A(x, \mathbf{e}(b_1^{n-2}), b_1^{n-2})[= x]$ *and*

(ii) $A(a_1^{n-2}, \mathbf{e}(a_1^{n-2}), x) = A(x, b_1^{n-2}\mathbf{e}(b_1^{n-2}))[= x]$.

The sketch of the proof of (i): 1) for $n = 2$ the formula (a) reduces to the formula $A(\mathbf{e}(\emptyset), x) = A(x, \mathbf{e}(\emptyset))$, and

2) Let $n \geq 3$. $F(x, b_1^{n-2}) \overset{def}{=} A(x, \mathbf{e}(b_1^{n-2}), b_1^{n-2}) \Rightarrow$

$$A(F(x, b_1^{n-2}), \mathbf{e}(b_1^{n-2}), b_1^{n-2}) = A(A(x, \mathbf{e}(b_1^{n-2}), b_1^{n-2}), \mathbf{e}(b_1^{n-2}), b_1^{n-2}) \Rightarrow$$
$$A(F(x, b_1^{n-2}), \mathbf{e}(b_1^{n-2}), b_1^{n-2}) = A(x, A(\mathbf{e}(b_1^{n-2}), b_1^{n-2}, \mathbf{e}(b_1^{n-2})), b_1^{n-2}) \Rightarrow$$
$$A(F(x, b_1^{n-2}), \mathbf{e}(b_1^{n-2}), b_1^{n-2}) = A(x, \mathbf{e}(b_1^{n-2}), b_1^{n-2}) \Rightarrow$$
$$F(x, b_1^{n-2}) = x \Rightarrow A(x, \mathbf{e}(b_1^{n-2}), b_1^{n-2}) = A(\mathbf{e}(a_1^{n-2}), a_1^{n-2}, x).$$

2.2 Remark: Let $n \geq 2$, let (Q,A) be an $n-$groupoid and let $\boldsymbol{\alpha}$ be an $(n-2)-$ary operation in Q. Then, for example, each of the following formulas

(1) $(\forall a_i \in Q)_1^{n-2}(\forall b_i \in Q)_1^{n-2}(\forall x \in Q)A(x, a_1^{n-2}, \boldsymbol{\alpha}(a_1^{n-2})) = A(b_1^{n-2}, \boldsymbol{\alpha}(b_1^{n-2}), x)$,

(2) $(\forall a_i \in Q)_1^{n-2}(\forall b_i \in Q)_1^{n-2}(\forall x \in Q)A(x, \boldsymbol{\alpha}(a_1^{n-2}), a_1^{n-2}) = A(\boldsymbol{\alpha}(b_1^{n-2}), b_1^{n-2}, x)$,

(3) $(\forall a_i \in Q)_1^{n-2}(\forall b_i \in Q)_1^{n-2}(\forall x \in Q)A(x, a_1^{n-2}, \boldsymbol{\alpha}(a_1^{n-2})) = A(\boldsymbol{\alpha}(b_1^{n-2}), b_1^{n-2}, x)$,

for $n = 2$ reduces to the formula

(4) $(\forall x \in Q)A(x, \boldsymbol{\alpha}(\emptyset)) = A(\boldsymbol{\alpha}(\emptyset), x)$.

Whence, we conclude that $\boldsymbol{\alpha}$, defined by any of the formulas (1) - (3), represents a generalization of the nullary operation - fixing a **central element** of the groupoid (Q,A).

2.3. Proposition [10]: *Let* $n \geq 2$, *let* (Q,A) *be an* $n-$*groupoid and let* $\boldsymbol{\alpha}$ *be an* $(n-2)-$*ary operation in* Q. *Then the following implications hold*

$(1) \Rightarrow (2) \wedge (3)$ *and* $(2) \Rightarrow (1) \wedge (3)$,

where (1) - (3) are statements from 2.2

2.4. Proposition [10]: *These is an algebra* $(Q, \{A, \boldsymbol{\alpha}\})$ *of the type* $\langle n, n-2 \rangle$ *such that the following holds: a)* (Q,A) *is an* $n-$*group , b)* $n \geq 3$, *c) the statement (3) from 2.2 and d) the statement (1) from 2.2 does not hold. [(2)* $\Longleftrightarrow$ *(1); 2.3].*

2.5. Proposition [10]: *Let $n \geq 3$, let (Q, A) be an $n-$group and let $\boldsymbol{\alpha}$ be an $(n-2)-$ary operation in Q. Further on, let the statement (1) [or statement (2)] from 2.2 [: 2.3] hold in the algebra $(Q, \{A, \boldsymbol{\alpha}\})$ [of the type $\langle n, n-2\rangle$]. Then there is a permutation α of the set Q such that for every $x \in Q$, for every sequence x_1^n over Q, for every sequence a_1^{n-2} over Q and for every $i \in \{1, ..., n\}$, the following equalities hold*

$$A(x, a_1^{n-2}, \boldsymbol{\alpha}(a_1^{n-2})) = \alpha(x) \quad \textit{and} \quad \alpha A(x_1^n) = A(x_1^{i-1}, \alpha(x_i), x_{i+1}^n).$$

[Whence, since $\alpha \in Q!$, we conclude that for every $i \in \{1, ..., n\}$ and for every $x_1^n \in Q$ also $A(x_1^{i-1}, \alpha^{-1}(x_i), x_{i+1}^n) = \alpha^{-1}A(x_1^n)$.]

2.6. Definition: [10]: *Let (Q, A) be an $n-$group and $n \geq 2$. Let also $\boldsymbol{\alpha}$ be an $(n-2)-$ary operation in the set Q. We say that $\boldsymbol{\alpha}$ is a* **central operation of the $n-$group** (Q, A) *iff the following formula holds*

$$(\forall a_i \in Q)_1^{n-2}(\forall b_i \in Q)_1^{n-2}(\forall x \in Q) \quad A(\boldsymbol{\alpha}(a_1^{n-2}), a_1^{n-2}, x) = A(x, \boldsymbol{\alpha}(b_1^{n-2}), b_1^{n-2})$$

[: formula (2) from 2.2].

2.7. Remark: The $\{1, n\}-$neutral operation $\mathbf{e}$ of the $n-$group (Q, A) is a central operation of that $n-$group [: 2.6, 2.1].

2.8. Proposition [4,5]:[1)] *Let $n \geq 3$, let (Q, A) be a $\langle 1, 2\rangle-$associative $n-$groupoid and let E be an $(n-2)-$ary operation in Q. In addition, let for every $x \in Q$, for every sequence a_1^{n-2} over Q and for every sequence b_1^{n-2} over Q the following equalities hold*

(a) $A(x, a^{n-2}, \mathsf{E}(a_1^{n-2})) = x$ *and*

(b) $A(b_1^{n-2}, \mathsf{E}(b_1^{n-2}), x) = x.$

Then (Q, A) is an $n-$group.

The sketch of the proof:

1^0 $(\forall a_i \in Q)_1^{n-2}(\forall a \in Q)(\forall x \in Q)(\forall y \in Q)(A(x, a, a_1^{n-2}) = A(y, a, a_1^{n-2}) \Rightarrow x = y)$. $[A(x, a, a_1^{n-2}) = A(y, a, a_1^{n-2}) \Rightarrow A(A(x, a, a_1^{n-2}), \mathsf{E}(a_1^{n-2}), c_1^{n-3}, \mathsf{E}(a, c_1^{n-3})) = A(A(y, a, a_1^{n-2}), \mathsf{E}(a_1^{n-2}), c_1^{n-3}, \mathsf{E}(a, c_1^{n-3})) \Rightarrow A(x, A(a, a_1^{n-2}, \mathsf{E}(a_1^{n-2})), c_1^{n-3}, \mathsf{E}(a, c_1^{n-3})) = A(y, A(a, a_1^{n-2}, \mathsf{E}(a_1^{n-2})), c_1^{n-3}, \mathsf{E}(a, c_1^{n-3})) \Rightarrow A(x, a, c_1^{n-3}, \mathsf{E}(a, c^{n-3})) = A(y, a, c^{n-3}, \mathsf{E}(a, c^{n-3})) \Rightarrow x = y$; (a), $n \geq 3$.]

2^0 (Q, A) is an $n-$semigroup. $[A(A(x_1^n), x_{n+1}^{2n-1}) = A(x_1, A(x_2^{n+1}), x_{n+2}^{2n-1})$, 1^0, 1.8; $n \geq 3$.]

3^0 $(\forall a \in Q)(\forall c_i \in Q)_1^{n-3} a = \mathsf{E}(c_1^{n-3}, \mathsf{E}(a, c_1^{n-3}))$. $[A(a, c_1^{n-3}, \mathsf{E}(a, c_1^{n-3}), \mathsf{E}(c_1^{n-3}, \mathsf{E}(a, c_1^{n-3}))) = \mathsf{E}(c_1^{n-3}, \mathsf{E}(a, c_1^{n-3}))$, $A(a, c_1^{n-3}, \mathsf{E}(a, c_1^{n-3}), \mathsf{E}(c_1^{n-3}, \mathsf{E}(a, c_1^{n-3}))) = a$; (b), (a), $n \geq 3$.]

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1) See: Corollary 5 in [4] and Theorem 2.6 in [5].

4^0 $(\forall a_i \in Q)_1^{n-2}(\forall a \in Q)(\forall x \in Q)(\forall y \in Q)(A(a, x, a_1^{n-2}) = A(a, y, a_1^{n-2}) \Rightarrow x = y)$. $[A(a, z, a_1^{n-2}) = A(\mathsf{E}(c_1^{n-3}, \mathsf{E}(a, c_1^{n-3})), z, a_1^{n-2}), A(a, x, a_1^{n-2}) = A(a, y, a_1^{n-2}) \Rightarrow A(c_1^{n-3}, \mathsf{E}(a, c_1^{n-3}), A(\mathsf{E}(c_1^{n-3}, \mathsf{E}(a, c_1^{n-3})), x, a_1^{n-2}), \mathsf{E}(a_1^{n-2})) = A(c_1^{n-3}, \mathsf{E}(a, c_1^{n-3}), A(\mathsf{E}(c_1^{n-3}, \mathsf{E}(a, c_1^{n-3})), y, a_1^{n-2}), \mathsf{E}(a_1^{n-2}))$; 3^0, 2^0 (: $\langle 1, n-1\rangle$–associative law), (b), (a)].

5^0 $(\forall a_i \in Q)_1^{n-2}(\forall a \in Q)(\forall x \in Q)(\forall y \in Q)(A(a_1^{n-2}, x, a) = A(a_1^{n-2}, y, a) \Longleftrightarrow x = y)$. $[A(a_1^{n-2}, x, a) = A(a_1^{n-2}, y, a) \Rightarrow A(d_1^2, A(a_1^{n-2}, x, a), d_3^{n-1}) = A(d_1^2, A(a_1^{n-2}, y, a), d_3^{n-1}) \Rightarrow A(A(d_1^2, a_1^{n-2}), x, a, d_3^{n-1}) = A(A(d_1^2, a_1^{n-2}), y, a, d_3^{n-1}) \Rightarrow x = y; x = y \Rightarrow A(a_1^{n-2}, x, a) = A(a_1^{n-2}, y, a)$; 2^0 (: $\langle 1, 3\rangle-$associative law), 4^0, $(n-1)-$monotony.]

6^0 $(\forall x \in Q)(\forall a \in Q)(\forall b \in Q)(\forall a_i \in Q)_1^{n-2}(A(a, x, a_1^{n-2}) = b \Longleftrightarrow x = A(c_1^{n-3}, \mathsf{E}(a, c_1^{n-3}), b, \mathsf{E}(a_1^{n-2})))$. $[A(a, x, a_1^{n-2}) = b \Longleftrightarrow A(c_1^{n-3}, \mathsf{E}(a, c_1^{n-3}), A(\mathsf{E}(c_1^{n-3}, \mathsf{E}(a, c_1^{n-3}), x, a_1^{n-2}), \mathsf{E}(a_1^{n-2})) = A(c_1^{n-3}, \mathsf{E}(a, c_1^{n-3}), b, \mathsf{E}(a_1^{n-2}))$; 3^0, 2^0 (: $\langle 1, n-1\rangle$-associative law), (b), (a).]

Finally, considering $2^0, 4^0$ and 6^0, by Proposition 1.2, we conclude that (Q, A) is $n-$group. □

Similarly, one could prove also the following proposition:

2.9. Proposition: *Let $n \geq 3$, let (Q, A) be an $\langle n-1, n\rangle-$associative $n-$groupoid and let E be an $(n-2)-$ary operation in Q. In addition, let for every $x \in Q$, for every sequence a_1^{n-2} over Q and for every sequence b_1^{n-2} over Q the following equalities hold*

$A(\mathsf{E}(a_1^{n-2}), a_1^{n-2}, x) = x$ *and* $A(x, \mathsf{E}(b_1^{n-2}), b_1^{n-2}) = x$.

Then (Q, A) is an $n-$group.

2.10. Remark: E from 2.8 and from 2.9 is an $\{1, n\}-$neutral operation of the $n-$group (Q, A) [: 2.8 (2.9), 1.5, 1.1].

3. Results

3.1. Theorem: *Let $n \geq 3$ and let (Q, A) be an $n-$groupoid. Then: (Q, A) is an $n-$group iff there are mappings $\boldsymbol{\alpha}$ and β, respectively, of the sets Q^{n-2} and Q into the set Q such that the laws*

(1) $A(A(x_1^n), x_{n+1}^{2n-1}) = A(x_1, A(x_2^{n+1}), x_{n+2}^{2n-1})$,

(2) $A(x, a_1^{n-1}, \boldsymbol{\alpha}(a_1^{n-2})) = A(b_1^{n-2}, \boldsymbol{\alpha}(b_1^{n-2}), x)$,

(3) $\beta A(x, c_1^{n-2}, \boldsymbol{\alpha}(c_1^{n-2})) = x$ *and*

(4) $\beta A(x_1^n) = A(x_1^{n-1}, \beta(x_n)) = A(x_1^{n-2}, \beta(x_{n-1}), x_n)$

hold in the algebra $(Q, \{A, \boldsymbol{\alpha}, \beta\})$.

Proof. *a)* $\Rightarrow$: Let (Q, A) be an $n-$group and let $\mathbf{e}$ be its $\{1, n\}-$neutral operation; $n \geq 3$. Whence, by Proposition 2.1 [(ii)], we conclude that there is [at least one] $(n-2)$-ary operation $\boldsymbol{\alpha}$ [$= \mathbf{e}$] and [at least one] unary operation

β $[= \{(x,x) \mid x \in Q\}]$ such that the laws (1) - (4) hold in the algebra $(Q, \{A, \boldsymbol{\alpha}, \beta\})$.

b) $\Leftarrow$: Let $(Q, \{A, \boldsymbol{\alpha}, \beta\})$ be an algebra of the type $\langle n, n-2, 1\rangle$ in which the laws (1) - (4) hold. By the assumption that in $(Q, \{A, \boldsymbol{\alpha}, \beta\})$ the laws (2) - (4) hold, we conclude that in $(Q, \{A, \boldsymbol{\alpha}, \beta\})$ also the following laws hold

(5) $A(x, a_1^{n-2}, \beta\boldsymbol{\alpha}(a_1^{n-2})) = x$ and

(6) $A(b_1^{n-2}, \beta\boldsymbol{\alpha}(b_1^{n-2}), x) = x$.

Since the laws (1), (5) and (6) hold in $(Q, \{A, \boldsymbol{\alpha}, \beta\})$, by Proposition 2.8, we conclude that (Q, A) is an $n-$group. [In addition, the $\{1, n\}-$neutral operation is defined by the formula

$$(\forall c_i \in Q)_1^{n-2} \mathbf{e}(c_1^{n-2}) = \beta\boldsymbol{\alpha}(c_1^{n-2});$$

2.10.]

Similarly, it is possible to prove that the following proposition holds:

3.2. Theorem: *Let $n \geq 3$ and let (Q, A) be an $n-$groupoid. Then: (Q, A) is an $n-$group iff there are mappings $\boldsymbol{\alpha}$ and β, respectively, of the sets Q^{n-2} and Q into the set Q such that the laws*

$(\bar{1})$ $A(x_1^{n-2}, A(x_{n-1}^{2n-2}), x_{2n-1}) = A(x_1^{n-1}, A(x_n^{2n-1}))$,

$(\bar{2})$ $A(\boldsymbol{\alpha}(a_1^{n-2}), a_1^{n-2}, x) = A(x, \boldsymbol{\alpha}(b_1^{n-2}), b_1^{n-2})$,

$(\bar{3})$ $\beta A(\boldsymbol{\alpha}(c_1^{n-2}), c_1^{n-2}, x) = x$ *and*

$(\bar{4})$ $\beta A(x_1^n) = A(\beta(x_1), x_2^n) = A(x_1, \beta(x_2), x_3^n)$

hold in the algebra $(Q, \{A, \boldsymbol{\alpha}, \beta\})$.

3.3. Theorem: *Let $n \geq 3$ and let (Q, A) be an $n-$group. Then (Q, A) is an $n-$group iff there are mappings $\boldsymbol{\alpha}$ and β, respectively, of the sets Q^{n-2} and Q into the set Q such that the laws*

$(\hat{1})$ $A(A(x_1^n), x_{n+1}^{2n-1}) = A(x_1, A(x_2^{n+1}), x_{n+2}^{2n-1})$ [*or*

$(\hat{\hat{1}})$ $A(x_1^{n-2}, A(x_{n-1}^{2n-2}), x_{2n-1}) = A(x_1^{n-1}, A(x_n^{2n-1}))]$,

$(\hat{2})$ $A(x, a_1^{n-2}, \boldsymbol{\alpha}(a_1^{n-2})) = A(\boldsymbol{\alpha}(b_1^{n-2}), b_1^{n-2}, x)$,

$(\hat{3})$ $\beta A(x, a^{n-2}, \boldsymbol{\alpha}(c_1^{n-2})) = x$ *and*

$(\hat{4})$ $\beta A(x_1^n) = A(x_1^{n-1}, \beta(x_n)) = A(\beta(x_1), x_2^n)$

hold in the algebra $(Q, \{A, \boldsymbol{\alpha}, \beta\})$.

Proof. $\hat{a}$) $\Rightarrow$: Let (Q, A) be an $n-$group and let $\mathbf{e}$ be its $\{1, n\}-$neutral operation [: 1.5] ; $n \geq 3$. Whence, by 1.3, we conclude that there is at least one] $(n-2)$- ary operation $\boldsymbol{\alpha}$ $[= \mathbf{e}]$ and at least one unary operation β $[= \{(x,x) | x \in Q\}]$ such that the algebra $(Q, \{A, \boldsymbol{\alpha}, \beta\})$ the laws $(\hat{1}), (\hat{\hat{1}}) - (\hat{4})$ hold.

$\hat{b})$ $\Leftarrow$: Let $(Q,\{A,\boldsymbol{\alpha},\beta\})$ be an algebra of the type $\langle n,n-2,1\rangle$ in which the laws $(\hat{1}),(\hat{2})-(\hat{4})[(\hat{\hat{1}}),(\hat{2})-(\hat{4})]$ hold. By the assumption that in $(Q,\{A,\boldsymbol{\alpha},\beta\})$ hold the laws $(\hat{2})-(\hat{4})$, we conclude that in $(Q,\{A,\boldsymbol{\alpha},\beta\})$ also the following laws hold

$(\hat{5})$ $A(x,a_1^{n-2},\beta\boldsymbol{\alpha}(a_1^{n-2}))=x$, and

$(\hat{6})$ $A(\beta\boldsymbol{\alpha}(b_1^{n-2}),b_1^{n-2},x)=x$

[either in $(Q,\{A,\boldsymbol{\alpha},\beta\})$ holds the laws $(\hat{1})$ or the law $(\hat{\hat{1}})$]. Since in $(Q,\{A,\boldsymbol{\alpha},\beta\})$ hold the laws $(\hat{1}),(\hat{5})$ and $(\hat{6})[(\hat{\hat{1}}),(\hat{5})$ and $(\hat{6})]$, by 1.6 and 1.7 we conclude that (Q,A) is an $n-$group. [The $\{1,n\}-$neutral operation $\mathbf{e}$ of the $n-$group (Q,A) satisfies the formula

$(\forall c_i\in Q)_1^{n-2}\mathbf{e}(c_1^{n-2})=\beta\boldsymbol{\alpha}(c_1^{n-2})$;

2.10.]

3.4. Example: Let $(\{1,2,3,4\},\cdot)$ be the Klein's group [Tab. 1] and $^{-1}$ the corresponding inverse operation. Further on, let φ be the permutation of the set $\{1,2,3,4\}$ defined in the following way

$$\varphi \overset{def}{=} \begin{pmatrix}1&2&3&4\\1&2&4&3\end{pmatrix}.$$

$\cdot$	1	2	3	4
1	1	2	3	4
2	2	1	4	3
3	3	4	1	2
4	4	3	2	1

Tab. 1

[Then: (a) $\varphi\in Aut(\{1,2,3,4\},\cdot)$; (b) $(\forall x\in\{1,2,3,4\})\varphi^2(x)=x$; (c) $\varphi(2)=2$; and (d) $\varphi(1)=1$].

3.4.1. Example: Let $A(x_1^3)\overset{def}{=}x_1\cdot\varphi(x_2)\cdot x_3\cdot 2$ and $\boldsymbol{\alpha}(c)\overset{def}{=}3\cdot(\varphi(c))^{-1}$. Then: (i) $(\{1,2,3,4\},A)$ is an 3-group; and (ii) for every $c\in\{1,2,3,4\}$ the following equalities hold $A(\boldsymbol{\alpha}(c),c,x)=4x$, $A(x,c,\boldsymbol{\alpha}(c))=4x$, $A(x,\boldsymbol{\alpha}(c),c)=3x$.
[See Proposition 2.4.]

3.4.2. Example: Let $B(x_1^3)\overset{def}{=}x_1\cdot\varphi(x_2)\cdot x_3$ and $\boldsymbol{\beta}(c)\overset{def}{=}2\cdot(\varphi(c))^{-1}$. Then: (1) $(\{1,2,3,4\},B)$ is an 3-group; and (2) for every $c\in\{1,2,3,4\}$ the following equalities hold $B(\boldsymbol{\beta}(c),c,x)=2x$ and $B(x,\boldsymbol{\beta}(c),c)=2x$.

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